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## LETTER TO THE EDITOR

## New families of isospectral hydrogen-like potentials

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**Abstract.** By applying algebraic techniques, we construct a two-parametric family of strictly isospectral hydrogen-like potentials, as well as some of its one-parametric limits. An additional one-parametric almost isospectral family of hydrogen-like potentials is also investigated. It is argued that the construction of a SUSY partner Hamiltonian using a factorization energy  $\delta$  less than the ground-state energy of the departure Hamiltonian is unnecessarily restrictive.

A considerable amount of work exists relating to exactly solvable one-dimensional potentials in quantum mechanics (QM). The main interest has been to enlarge the number of analytically solvable potentials using diverse techniques: the Darboux transformation [1], the Gelfand–Levitan formalism [2], the standard and modified factorizations [3,4], the supersymmetric (SUSY) QM [5] etc. The underlying ideas of most of these procedures have been summarized in an algebraic scheme in which a first-order differential operator intertwines two different Hamiltonians [6]. In particular, the generation of one-parametric families of potentials isospectral to the traditional ones [2–9] can be performed by means of this *first-order intertwining technique*. This method also leads naturally to the factorization of the implied Hamiltonians.

Recently, a generalization of this technique, in which the intertwining operator is of second order, has been used to derive a two-parametric family of potentials isospectral to the harmonic oscillator [10]. This *second-order intertwining technique* (SOIT) is a particular case with  $n = 2$  of the  $n$ -order intertwining technique which appears to be the best way in which to introduce the higher order SUSY QM. In principle, the latter makes it possible to generate  $n$ -parametric families of Hamiltonians isospectral to a given Hamiltonian [11–13].

In this paper we shall generate a two-parametric family of radial isospectral hydrogen-like potentials by means of the SOIT. We shall also show that the one-parametric family of potentials derived by Fernández [8] can be recovered from ours. As a final result, we shall find a one-parametric family of potentials having the same energy levels as the corresponding radial hydrogen-like potentials except for the ground-state energy level.

The standard procedure used to deal with hydrogen-like potentials in QM reduces to solve the eigenproblem for a particle in a one-dimensional effective potential  $V_l(r) = l(l + 1)/r^2 - 2/r$ , where  $l = 0, 1, 2, \dots$  is the azimuthal quantum number and  $r$  is a dimensionless radial coordinate. For simplicity, instead of working with the standard radial wavefunctions  $R(r)$ , we shall work with the functions  $\psi(r) \equiv rR(r)$  with an inner product

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defined by  $\langle \psi, \psi' \rangle \equiv 4\pi \int_0^{+\infty} \bar{\psi}(r)\psi'(r) dr$ . It is well known that the eigenvalues of the radial Hamiltonian  $H_l = -d^2/dr^2 + V_l(r)$  (with fixed  $l$ ) are given by

$$E_n \equiv E_{lk} = -\frac{1}{(l+k)^2} \quad k = 1, 2, 3, \dots \quad (1)$$

where  $l+k=n$ .

We are looking for a Hamiltonian  $\tilde{H}_{l'} = -d^2/dr^2 + \tilde{V}_{l'}(r)$ , such that the following intertwining relationship is satisfied [6]:

$$\tilde{H}_{l'} A = A H_l \quad (2)$$

where  $A$  is a second-order differential operator to be determined

$$A \equiv \frac{d^2}{dr^2} + \beta(r) \frac{d}{dr} + \gamma(r). \quad (3)$$

Equations (2) and (3) and the explicit form of  $H_l$  and  $\tilde{H}_{l'}$  lead to the following relations between  $V_l(r)$ ,  $\tilde{V}_{l'}(r)$ ,  $\beta(r)$  and  $\gamma(r)$ :

$$\beta\beta'' - \frac{\beta^2}{2} + \left(2\gamma(r) - \beta' - \frac{\beta^2}{2}\right)\beta^2 + 2c = 0 \quad (4)$$

$$\tilde{V}_{l'}(r) = \frac{l(l+1)}{r^2} - \frac{2}{r} + 2\beta' \quad (5)$$

$$2\gamma(r) = \beta^2 - \beta' - 2\frac{l(l+1)}{r^2} + \frac{1}{r} - d \quad (6)$$

where  $c$  and  $d$  are, in principle, arbitrary constants and the prime denotes a derivative with respect to  $r$ . The key point now becomes solving the nonlinear second-order differential equation (4) for  $\beta(r)$ . First let us enforce that the operator  $A$  includes  $a_{l-1}a_l$  as a particular case, where  $a_l = (-d/dr + l/r - 1/l)$  is the standard factorization operator for the radial hydrogen-like Hamiltonians [3]. This condition permits us, without a loss of generality, to fix the constants  $c$  and  $d$  in (4)–(6) as

$$c = \frac{(2l-1)^2}{4l^4(l-1)^4} \quad d = \frac{1+(2l-1)^2}{2l^2(l-1)^2}. \quad (7)$$

After including (7) in (4) and (6), it turns out that the general solution to (4) becomes

$$\beta(r) = \frac{1-2l}{l^2(l-1)^2} \left[ \frac{d}{dr} \ln \left( \frac{g_2(r)}{g_1(r)} \right) \right]^{-1} \quad l = 2, 3, \dots \quad (8)$$

$$g_1(r) = \left\{ 1 - \frac{v_1}{(2l)!} \left( \frac{2}{l} \right)^{2l+1} \int_0^r x^{2l} e^{-2x/l} dx \right\} \quad (9)$$

$$g_2(r) = e^{r/l(l-1)} \left[ 1 - \frac{r}{l(l-1)} \right] \left\{ 1 + \frac{v_2}{(2l-1)!} \left( \frac{2}{l-1} \right)^{2l-1} \int_0^r \frac{x^{2l} e^{-2x/(l-1)}}{[l(l-1)-x]^2} dx \right\} \quad (10)$$

where  $v_1$  and  $v_2$  are integration constants.

The asymptotic behaviour of  $\beta(r)$  is given by  $\beta(r) \sim (1-2l)/l(l-1)$ , while it diverges as  $(2l-1)/r$ , when  $r \rightarrow 0^+$ . This suggests to us to write the new potential  $\tilde{V}_{l'}(r)$  in (5) as

$$\tilde{V}_{l-2}(r) = V_{l-2}(r) + 2\alpha'(r) \quad l = 2, 3, \dots \quad (11)$$

where  $l' = l-2$  and  $\alpha(r) \equiv \beta(r) + (1-2l)/r$ , is an appropriate function that makes evident the limit  $\tilde{V}_{l-2}(r) \rightarrow V_{l-2}(r)$ , when  $r \rightarrow +\infty$  and  $r \rightarrow 0$ . The parameter domain for which  $\alpha'(r)$  is free of singularities is given by  $v_1, v_2 \in (-\infty, 1)$ ; inside this parameter region, the new two-parametric family of potentials  $\tilde{V}_{l-2}(r)$  has the same singularities as

$V_{l-2}(r)$ . Furthermore, for  $\nu_1 = \nu_2 = 0$  we have  $\alpha' = 0$  and  $\tilde{V}_{l-2}(r) = V_{l-2}(r)$ . This means that  $V_{l-2}(r)$  not only governs the asymptotic behaviour of  $\tilde{V}_{l-2}(r)$  but it is also a member of the family (11).

Now, from equation (2) it becomes apparent that the operator  $A$  acting on the eigenfunctions  $\{\psi_{nl}(r)\}$  of  $H_l$  provides eigenfunctions  $\tilde{\psi}_{n,l-2}(r) \propto A\psi_{nl}(r)$ ,  $l = 2, 3, \dots$  of  $\tilde{H}_{l-2}$  with eigenvalues  $\tilde{E}_{lk} = E_{lk}$ , i.e.

$$\tilde{\psi}_{n,l-2}(r) = \frac{l(l-1)n^2}{\sqrt{(n^2-l^2)(n^2-l^2+2l-1)}} A\psi_{nl}(r). \tag{12}$$

However, the set  $\{\tilde{\psi}_{n,l-2}(r)\}$  is not yet complete in the Hilbert space of square-integrable functions  $\mathcal{H}$ . This is clear if we try to answer the following questions: are there functions  $\tilde{\psi}_{l-2,\epsilon}(r)$  orthogonal to all the  $\tilde{\psi}_{n,l-2}(r)$ ? If so, are they eigenfunctions of  $\tilde{H}_{l-2}$ ? In order to answer these questions, let us assume that the set  $\{\tilde{\psi}_{l-2,\epsilon}\}$  exists, then

$$\langle \tilde{\psi}_{l-2,\epsilon}(r), \tilde{\psi}_{n,l-2}(r) \rangle \propto \langle \tilde{\psi}_{l-2,\epsilon}(r), A\psi_{nl}(r) \rangle = \langle A^\dagger \tilde{\psi}_{l-2,\epsilon}(r), \psi_{nl}(r) \rangle = 0.$$

Owing to the fact that  $\{\psi_{nl}(r)\}$  is a complete set in  $\mathcal{H}$ , the kernel of the second-order differential operator  $A^\dagger$  is a two-dimensional subspace  $\mathcal{H}_\epsilon$  orthogonal to all the  $\tilde{\psi}_{n,l-2}(r)$ ,  $l = 2, 3, \dots$ . Let us write them as  $\tilde{\psi}_{l-2,\epsilon}(r) = c_0 \exp[\int f(x) dx]$ , where  $c_0$  is a constant and  $f(x)$  is to be determined. The equation  $A^\dagger \tilde{\psi}_{l-2,\epsilon}(r) = 0$  can be rewritten as

$$f'(r) - \beta(r)f(r) + f^2(r) - \beta'(r) + \gamma(r) = 0. \tag{13}$$

This Riccati-type differential equation has a general solution given by

$$f(r) = \frac{1}{l} - \frac{l}{r} + \beta(r) + \frac{d}{dr} \ln[c_1 g_1(r) + c_2 g_2(r)] \tag{14}$$

where  $g_1(r)$  and  $g_2(r)$  are given in (9) and (10), and  $c_1$  and  $c_2$  are constants. The generic kernel element is given by  $\tilde{\psi}_{l-2,\epsilon}(r) = C_0 \tilde{\psi}_{l-2,0}(r) + C_{-1} \tilde{\psi}_{l-2,-1}(r)$ , where

$$\tilde{\psi}_{l-2,0}(r) = \sqrt{\left(\frac{1-\nu_1}{(2l)!}\right) \left(\frac{2}{l}\right)^{2l+1} (2l-1) \left(\frac{1}{l(l-1)}\right) \frac{r^l e^{-r/l} g_2(r)}{W(g_1, g_2)}} \quad l = 2, 3, \dots \tag{15}$$

and

$$\tilde{\psi}_{l-2,-1}(r) = \sqrt{\left(\frac{1-\nu_2}{2l(2l)!}\right) \left(\frac{2}{l-1}\right)^{2l+1} (2l-1) \left(\frac{1}{l(l-1)}\right) \frac{r^l e^{-r/l} g_1(r)}{W(g_2, g_1)}} \quad l = 2, 3, \dots \tag{16}$$

are both eigenfunctions of  $\tilde{H}_{l-2}$  with eigenvalues  $\tilde{E}_{l-2,0} = -1/l^2$ ,  $\tilde{E}_{l-2,-1} = -1/(l-1)^2$ ,  $l = 2, 3, \dots$  respectively. In (15) and (16),  $W(g_1, g_2) = g_1'(r)g_2(r) - g_1(r)g_2'(r)$  is the negative of the Wronskian of  $g_1(r)$  and  $g_2(r)$ .

Notice that  $\{\tilde{\psi}_{l-2,-1}, \tilde{\psi}_{l-2,0}, \tilde{\psi}_{n,l-2}, l = 2, 3, \dots\}$  is now a complete set in  $\mathcal{H}$ , and their elements are eigenfunctions of  $\tilde{H}_{l-2}$ ,  $l = 2, 3, \dots$ , with eigenvalues

$$\tilde{E}_{l-2,k} = -\frac{1}{(l-1)^2}, -\frac{1}{l^2}, -\frac{1}{(l+k)^2} \quad k = 1, 2, 3, \dots \tag{17}$$

Comparing with (1), one can see that this spectrum is identical to that of  $H_{l-2}$ , i.e.  $E_{l-2,k} = \tilde{E}_{l-2,k}$ . Hence, the Hamiltonian  $\tilde{H}_{l-2}$  is strictly isospectral to  $H_{l-2}$ , and because  $\tilde{V}_{l-2}(r)$  depends on two free parameters  $\nu_1$  and  $\nu_2$ , a new two-parametric family of isospectral hydrogen-like potentials has been generated. Some particular cases are worth discussing in more detail.

Firstly, for  $v_1 = v_2 = 0$ , the well known hydrogen-like potential  $V_{l-2}(r)$  is always recovered because it is a member of the family  $\tilde{V}_{l-2}(r)$ .

Let us now take  $v_1 = 0$  ( $v_2 = 0$ ), in this case the two-parametric family  $\tilde{V}_{l-2}(r)$  is a new one-parametric family which becomes a hydrogen-like potential when  $v_2 \rightarrow 0$  ( $v_1 \rightarrow 0$ ). In particular, let  $v_1 = 0$ , and

$$v_2 = \left(\frac{l-1}{2}\right)^{2l-1} \frac{(2l-2)!}{\gamma_{l-1}}.$$

In this case, the function  $\alpha(r)$  becomes

$$\alpha(r) = \frac{1-2l}{l(l-1)} + \frac{r^{2l-2}e^{-2r/(l-1)}}{\gamma_{l-1} - \int_0^r x^{2l-2}e^{-2x/(l-1)} dx} \quad l = 2, 3, \dots$$

In (11) if we change  $l$  by  $l+1$  with  $\alpha(r)$  as defined above, we obtain the following one-parametric family of potentials:

$$\tilde{V}_{l-1}(r) = V_{l-1}(r) + 2 \frac{d}{dr} \left\{ \frac{r^{2l}e^{-2r/l}}{\gamma_l - \int_0^r x^{2l}e^{-2x/l} dx} \right\} \quad l = 1, 2, \dots \quad (18)$$

which was generated by Fernández in 1984 [8]. Notice that, in the case when  $\gamma_l \rightarrow \frac{1}{4}$ , the family (18) gives the particular case derived by Abraham and Moses [7]. Moreover, when  $\gamma_l \rightarrow \infty$  ( $v_2 \rightarrow 0$ ), we have  $\tilde{V}_{l-1}(r) \rightarrow V_{l-1}(r)$ , just as we have proposed.

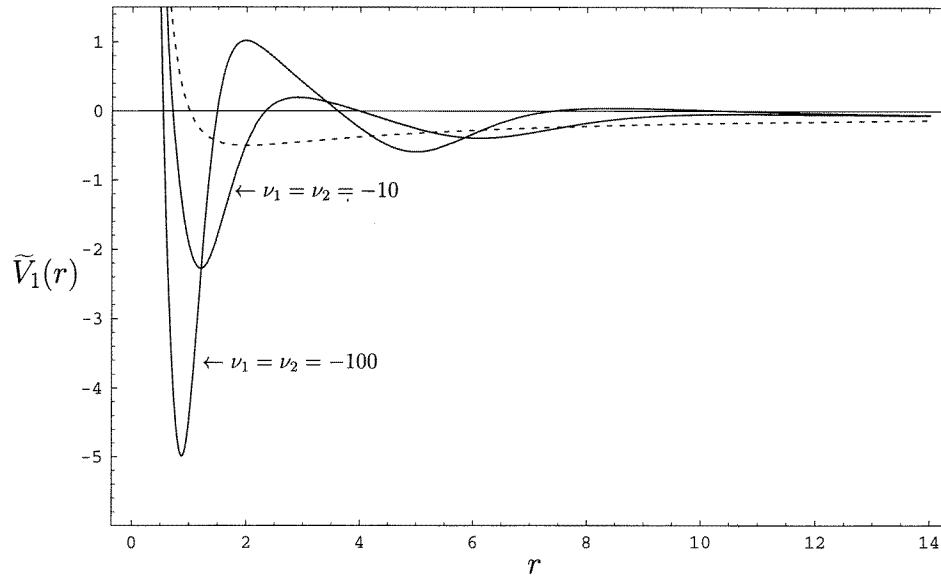
The previous potentials can be seen as deformations of  $V_{l-1}(r)$  induced by the second term in (18), which does not change the behaviour of  $V_{l-1}(r)$  at the ends of the interval  $[0, \infty)$ , but can produce important modifications inside. In particular, there is the possibility of creating one additional well in  $V_{l-1}(r)$  whose depth and position can be changed by varying  $\gamma_l$ . On the other hand, the two-parametric family of potentials (11) also admits the previous interpretation, but more freedom is given by the deforming term (it depends on two parameters instead of just one as in the previous case). Hence, we now find the possibility of introducing two wells, one of them with its minimum placed around the global minimum of  $V_{l-2}(r)$  and the other pushed further out. As in the previous one-parametric case, the depths and positions of the two wells can be modified by changing the two parameters  $v_1$  and  $v_2$ . This is illustrated in figure 1, where two members of the family  $\tilde{V}_1(r)$  of (11) are depicted, together with the undeformed potential  $V_1(r)$  (broken curve). In figure 2 we have also plotted the corresponding probability densities for the two energy levels  $\tilde{E}_{1,-1} = -\frac{1}{4}$ ,  $\tilde{E}_{1,0} = -\frac{1}{9}$  with  $v_1 = v_2 = -10$ . For the lowest level the probability has a maximum around the left well. On the other hand, the first excited state has two maxima, the highest one centred around the right well, while the lowest one is situated around the left well. When we go over the higher excited states, the probabilities resemble increasingly more the corresponding hydrogen-like densities.

Until now, departing from the hydrogen-like potential  $V_l(r)$  we have generated a two-parametric family of solvable potentials  $\tilde{V}_{l-2}(r)$ , with the same spectrum and singularities as  $V_{l-2}(r)$ . Now, as the intertwining operator  $A$  is of second order, it is interesting to look for its possible factorization in terms of two first-order differential operators  $b_1$ , and  $b_2$ , i.e.

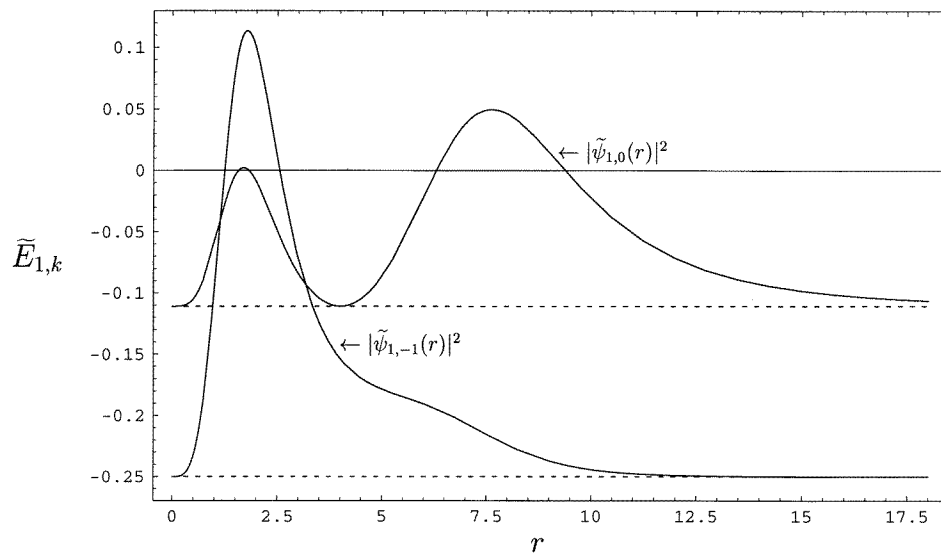
$$A = b_2 b_1 \quad b_j = \frac{d}{dr} + w_j(r) \quad j = 1, 2. \quad (19)$$

This leads to  $\gamma(r) = w_2' + w_1 w_2$ , and  $w_1(r) = \beta(r) - w_2(r)$ , where  $\beta(r)$  is given by (8), and  $w_2$  has the form given in (14). Hence,  $w_1(r)$  takes the form

$$w_1(r) = \frac{l}{r} - \frac{1}{l} - \frac{d}{dr} \ln[c_1 g_1(r) + c_2 g_2(r)]. \quad (20)$$



**Figure 1.** The hydrogen-like potential  $V_{l-2}(r)$  (broken curve) and two members of the family  $\tilde{V}_{l-2}(r)$  (full curve) with  $l = 3$ . Here we have chosen equal values for  $\nu_1$  and  $\nu_2$ , but it is not a restriction (see equation (11) and below). Note that in the limit case when  $\nu_i \rightarrow 0$ ,  $i = 1, 2$ , the full curves closely resemble the broken curve.



**Figure 2.** Behaviour of the probability densities of the two first energy levels  $-\frac{1}{4}$  and  $-\frac{1}{9}$  of  $\tilde{H}_1$ , with  $\nu_1 = \nu_2 = -10$ . The levels are indicated by the broken curves.

It is now clear that when solving (13), we have simultaneously obtained the solutions to the equation  $A^\dagger \tilde{\psi}_{l-2,\epsilon} = 0$ , as well as the factorizations of the operator  $A$ . There is a continuous family of factorizations because when we change the values of  $c_1$  and  $c_2$  in (20), we are simultaneously changing the operators  $b_1$  and  $b_2$ , but maintaining fixed their

product  $A = b_2 b_1$ .

The first consequence of this factorization arises after rewriting equation (2) as  $\tilde{H}_{l-2} b_2 b_1 = b_2 b_1 H_l$ . Now suppose that there is a Hamiltonian  $H_{l-1}^* = -d^2/dr^2 + V_{l-1}^*$  such that  $H_{l-1}^* b_1 = b_1 H_l$ . Thus  $b_2 H_{l-1}^* b_1 = b_2 b_1 H_l$ , and we obtain  $\tilde{H}_{l-2} b_2 = b_2 H_{l-1}^*$ . Therefore,  $H_{l-1}^*$  could be considered as an intermediate Hamiltonian between  $H_l$  and  $\tilde{H}_{l-2}$ . Hence, the SOIT can be seen as the iteration of two first-order intertwining transformations.

We note here that from the very beginning we are labelling with the subindex  $l-1$  the intermediate Hamiltonian  $H^*$ . This is a consequence of further calculations leading to a centrifugal term for  $V^*$  with exactly that index.

In order to ensure that the first-order intertwining relationship  $H_{l-1}^* b_1 = b_1 H_l$  would be satisfied, the functions  $w_1$  and  $V_{l-1}^*$  must satisfy some restrictions. The key one becomes the following Riccati equation:

$$-w_1' + (w_1)^2 - V_l + \delta_1 = 0 \quad (21)$$

complemented with the typical SUSY relationship  $V_{l-1}^* = V_l + 2w_1'$ , where  $\delta_1$  is a constant to be determined. We also note that a first-order intertwining relationship of the kind  $H_{l-1}^* b_1 = b_1 H_l$  leads naturally to the factorization of the Hamiltonians  $H_l$  and  $H_{l-1}^*$ :  $H_l = b_1^\dagger b_1 + \delta_1$ , and  $H_{l-1}^* = b_1 b_1^\dagger + \delta_1$ . By a similar argument, the Hamiltonians  $H_{l-1}^*$  and  $\tilde{H}_{l-2}$  become factorized in terms of  $b_2$  and  $b_2^\dagger$ :  $H^* = b_2^\dagger b_2 + \delta_2$ , and  $\tilde{H} = b_2 b_2^\dagger + \delta_2$ .

In order to determine the intermediate Hamiltonian  $H_{l-1}^*$ , we must find inside the family (20) a member obeying equation (21). In fact, for  $c_1 = 0$  and  $c_2 = 1$ , we have  $\delta_1 = -1/(l-1)^2$ ,  $l = 2, 3, \dots$ . Thus, the potential  $V_{l-1}^*(r)$  can be written as

$$V_{l-1}^*(r) = \frac{l(l-1)}{r^2} - \frac{2}{r} + 2 \left[ \frac{(g_2')^2 - g_2'' g_2}{(g_2)^2} \right]. \quad (22)$$

The parameter domain for which the family (22) has the same singularity as  $V_{l-1}(r)$  is given by  $v_2 \in (1, \infty)$ . The eigenfunctions of  $H_{l-1}^*$ ,  $l = 2, 3, \dots$  are given by  $\psi_{l-1,-1}^* \propto r^l e^{-r/l}/g_2$ , and  $\psi_{n,l-1}^* \propto b_1 \psi_{nl}$ , with eigenvalues  $E_{l-1,-1}^* = -1/(l-1)^2$ , and  $E_{l-1,k}^* = E_{lk}$ ,  $k = 1, 2, \dots$ , respectively. Note the unusual absence of the state corresponding to  $E_{l-1,0}^* = -1/l^2$ . A direct comparison of the spectra shows that  $V_{l-1}^*(r)$  is almost isospectral to  $V_{l-1}$ , the difference resting on the ground-state level position.

The next first-order intertwining transformation gives a different factorization of  $H_{l-1}^*$  and some interesting new results. The absent energy level  $\tilde{E}_{l-2,0} = E_{l-1,0}^*$  is now added to the spectrum of  $H_{l-1}^*$  in order to generate  $\tilde{H}_{l-2}$ . However, this means that the factorization energy in this second step is greater than the ground-state energy level of  $H_{l-1}^*$ , and this naturally fills the *hole* generated by the first factorization. The eigenfunctions of  $\tilde{H}_{l-2}$ , in terms of those of  $H_{l-1}^*$ ,  $l = 2, 3, \dots$ , are given by  $\{\tilde{\psi}_{l-2,-1} \propto b_2 \psi_{l-1,-1}^*, \tilde{\psi}_{l-2,0}, \tilde{\psi}_{n,l-2} \propto b_2 \psi_{n,l-1}^*\}$ , where  $\tilde{\psi}_{l-2,0} \propto \exp(-\int w_2(x) dx)$  is the eigenfunction associated with the 'missing' energy level  $E_{l-1,0}^* = \tilde{E}_{l-2,0}$ . A direct calculation shows that this set of eigenfunctions is the same as that derived by means of the SOIT.

*Concluding remarks.* In this paper we have shown that the SOIT allows one to derive a two-parametric family of isospectral hydrogen-like potentials. The iteration of two first-order intertwining transformations leads to the same results but gives additional information. Thus, against the standard statement of SUSY QM, there are cases where a factorization energy greater than the ground-state energy of the departure Hamiltonian leads to a physically acceptable SUSY partner. A deeper discussion of these first-order intertwining cases will

be given elsewhere [13]. Finally, all the potentials presented here have the same kind of singularity at a fixed point ( $r = 0$ ) as the initial hydrogen-like Hamiltonian. The case when the intertwined potentials are non-singular has been illustrated by Fernández *et al* for the harmonic oscillator [6, 10, 12], while the case when the SUSY partner of the oscillator potential has a movable singularity has been successfully interpreted [14]. The corresponding problem for a family of isospectral hydrogen-like potentials with a different singularity as those of  $V_l(r)$  in  $r = 0$  is open.

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